AMYGDALA $z \mapsto z^2 + c$

A Newsletter of Fractals & $\mathcal{M}$ (the Mandelbrot Set) Issue #25

July 18, 1991

IN THIS ISSUE...

The Amygdalan Sects, by John Dewey Jones, originally appeared in Amygdala #1, introducing a new genre, which one reviewer called *Math Fiction.* It is my favorite item from the 25 preceding issues, and rather than allow it to recede peacefully into the mists of time, I have resuscitated it in this issue. I hope that you, dear reader, will enjoy it half so much as I do, at least.

Journey To The Interior, by Martin Dace, discusses some ways to penetrate the heart of the darkness which seems to pervade the Mandelbrot set's interior, bringing its structure to the light of day.

The Secant Method, by John Dewey Jones (again!), analyzes and examines the computer-graphical beauties hidden within the secant method, a variant of Newton's method.

The lives, both real and imagined, of our two authors are curiously intertwined. For example, Dace makes a brief appearance as a character in *The Amygdalan Sects.* This intertwining is partially unraveled in their mini-biographies, given under *Authors.*

THE SLIDES (S25)

The four slides in the slide supplement to this issue show four of the ten images which illustrate *The Secant Method.* Since they are described there in some detail, I will merely list them here: the usual Amygdala series numbers followed by Jones's numbering:

1030: SR2

1031: SR3

1032: SC1

1032: SC2

Some of you will want to have the entire set of ten slides. I offer them both as a set of all ten, and as a set of those six of the ten that do not appear in slide set S25. See the current order form for prices.

THE AMYGDALAN SECTS

— *John Dewey Jones*

From the Encyclopædia Galactica, 19th Edition:

"The decline and eventual collapse of Earth's earliest space-faring civilizations had long mystified archaeologists. It was Dr. Martin Dace's work on the Amygdalan Sects which eventually provided an explanation. Dr. Dace was at that time a postgraduate student at the Theological Department of the University of Altair, and had become interested in the psychology of the many bizarre and irrational cults that had sprung up around the time of the collapse. That he picked the Amygdalans for his thesis topic was perhaps pure serendipity. But the skill which he brought to piecing together the strange history of the sect from its fragmentary records shows imaginative genius of the first order.

"Shortly before the collapse, human civilization had constructed its first computers. Crude by our

CONTENTS

own standards, these machines still used electronic circuitry as a basis for operation. Nevertheless, they were sufficiently powerful to provide glimpses of the M Object. The Object, infinitely complex and infinitely beautiful, acted like a lodestone to the finest minds on Earth at that time. They found in it deep mathematical theorems, hinting at insights into phenomena of the natural world; they found aesthetic satisfaction in contemplation of its intricate form; their curiosity led them to explore ever further into its infinite labyrinths. To this task they harnessed the most powerful tools available to them: the computers of industry, of government and of the war lords spent every spare moment focused on the Object.

"The more deeply the Object was investigated, the more alluring it became. Its investigators applied their ingenuity to securing more computer time and access to faster computers. Because the Object's appeal was greatest to the most imaginative and able members of terrestrial society, they were largely successful in gaining access to the resources they required. Openly or surreptitiously, a growing fraction of Earth's intellectual and electronic resources were committed to exploration of the Object. The effects of this were slow to be felt; the diversion of resources was greatest in areas least subject to outside supervision — academia, government laboratories and military research.

"The extent to which the Object had preoccupied Earth's information-processing systems became dramatically apparent in the 'Fizzle War' between the two major power-blocs on the planet at that time. The political leaders on either side had escalated through a series of ritual insults, provocative gestures and grandstanding to a level from which they could not back down. Having carefully checked the contingency plans for their own survival, they gave orders to commence global thermonuclear war.

"Nothing happened. The computers on both continents were locked in contemplation of the area around 0.0155-0.7i, where it was suspected that an intriguing new structure lurked. After several embarrassing hours, the politicians found a face-saving formula and backed off.

"This happy outcome, however, presaged a more complete breakdown of society's mechanisms. Computers had penetrated most of the manufacturing and service industries, and as their attention came increasingly to rest upon the Object, the fabric of the industrial civilizations came apart. This collapse fed upon itself, as more and more of those who might have averted disaster chose to turn their attention from the collapsing economy and garbage-strewn cities to the serene beauty of the Object. It was as though the electronic brains that supervised the world's maintenance systems had become absentminded; flickering traffic signals grid-locked the traffic in city streets; monitored reactors melted down or quietly switched themselves off.

"Then the ultimate calamity occurred. The collapsing infrastructure could no longer provide the steady diet of smoothed electrical power that the computers required. One by one, the windows looking in on the Object slammed shut.

"A profound sadness settled over the world. As a philosopher of a slightly earlier period had said, the loss of any pleasing object is painful; the loss of an infinitely pleasing Object must necessarily be infinitely painful. There were, of course, pictures that had been taken before the loss of contact, but these, finite, static, and limited, seemed only to mock the vibrant interaction that had been possible with the Object itself.

"This, then, was the picture Dr. Dace pieced together of the collapse. And this, too, was the origin of the Amygdalan Sects — the small groups that grew out of the collapse, following diverse routes but all motivated by a deep yearning for lost beauty. We append here an extract from Dace's thesis, describing the festival rite of one of the Sects:

"...on the Festival Days, the worshippers gather in the Central Park in the evening. Each, carrying his abacus and the Seven Parasols, goes to his preassigned spot; the priests assist any who are in difficulty. At sunset, the High Priest reads out the Objective Coordinates of the northeast and southwest corners of the park; his words are passed through the assembly by special messengers, and the congregation then sits down to compute. Each first computes his own Objective Coordinates, and then Iterates, intoning the mantra of the sect — the MO mantra. On reaching a result of magnitude greater than twice Unity, he sets aside his abacus and sits in silent meditation. From time to time the

priests pass among the congregation, checking the total iterations that each of the meditators has achieved.

"Just before dawn on the following day, all cease their iterations and wait. From the Tower of Objective Vision, the High Priest announces the numbers limiting the seven ranges. Then, just as the sun rises, each worshipper erects the parasol corresponding to his calculated number. The effect is incomparably beautiful, though it can only be appreciated fully from the top of the Tower..."

JOURNEY TO THE INTERIOR

— *Martin Dace*

Most images of $\mathcal{M}$ show the interior of the boundary as formless black. Nevertheless there is a very natural way of showing the interior, identical in principle with the usual method of showing the exterior.

Points exterior to $\mathcal{M}$ are most often coloured according to how long it takes their iterates to converge on an attractor. In the case of all such points the attractor is infinity. Points interior to $\mathcal{M}$, on the other hand, tend towards different attractors, some single and some periodic.

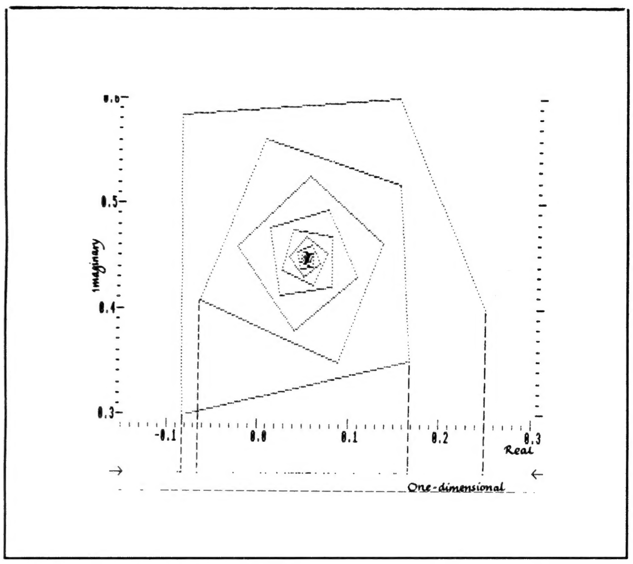


Figure 1

Let us look at the journeys taken by the iterates of individual points. A point outside $\mathcal{M}$ is shown spiralling away towards infinity in the tutorial in Amy #2. On the inside of the set the iterates spiral inwards, towards a single value (Figure 1) or towards a cycle of values. Once locked into a cycle, the iterates repeat the same sequence forever.

A small adjustment to the Mandelbrot set program will enable us to see how the process of spiralling towards a stable attractor changes as the imaginary part of the starting number c is changed (the formula being as usual: $z_{n+1} = z_n^2 + c$) We plot the iterates z_n for each c, but we turn the plot edge on, so that we see only from the side how the iterates converge on stable values (see bottom of Figure 1). Then we can stack up these one-dimensional views into a two-dimensional diagram. In other words, each iterate z_n is plotted as the point $(\Re z_n, \Im c)$[1]. Holding $\Re c$ fixed at 0.25, for example, and varying $\Im c$, we get a translucent tree (Figure 2).

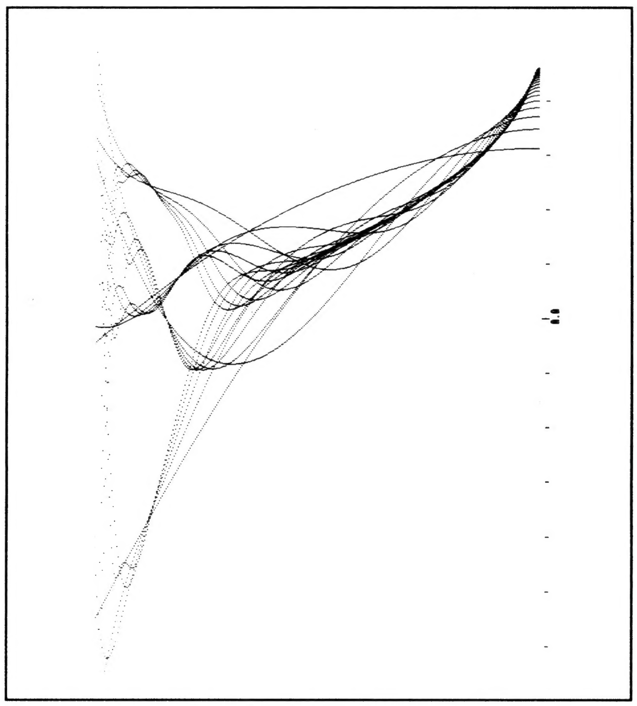


Figure 2 — Verhulst Process, Mandelbrot Variant for the Line $\Re c = 0.25$ (All Transients Plotted)

If we ignore the early iterates — the transients — we will see only the centre of this tree: the real

1. $\Re z_n$ = real part of z_n; $\Im c$ = imaginary part of c.

part of the attractor varying with the value of $\Im c$ (Figure 3).

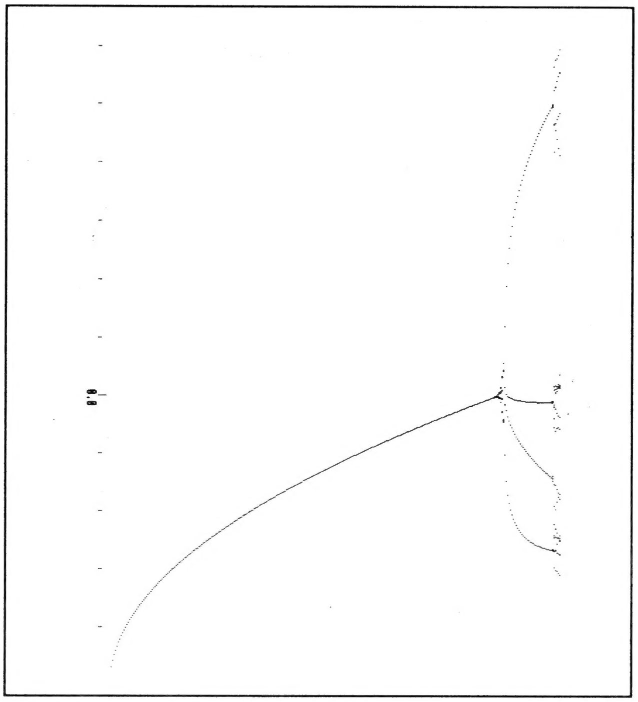


Figure 3 — Verhulst Process, Mandelbrot Variant for the Line $\Re c$ = 0.25 (500 Transients Omitted)

What happens at the point of branching? We see four braids winding out from a common stem, circling one another before separating out (Figure 4). The attractor now has a period of 4.

The nearer to the edge of the Mandelbrot set we go, the longer it takes for the transients to die away: for the iterates to approach their attractor. This is exactly what happens *outside* $\mathcal{M}$: the nearer c is to the edge of $\mathcal{M}$, the longer it takes for the iterates to take off towards the outside attractor: infinity.

To map this, I have used an idea given in the article that started us off on all this in the first place: A. K. Dewdney's column in Scientific American for August, 1985. This is the "rho" function or "slow/fast" trick cited by Rollo Silver as the basis for one of his "speedup" methods in Amygdala #1. What I have done is simply to colour each point in the interior of $\mathcal{M}$ according to how long it takes for the iterates to converge within a (very small) distance of their attractor — whether the attractor is single or periodic (Figure 5).

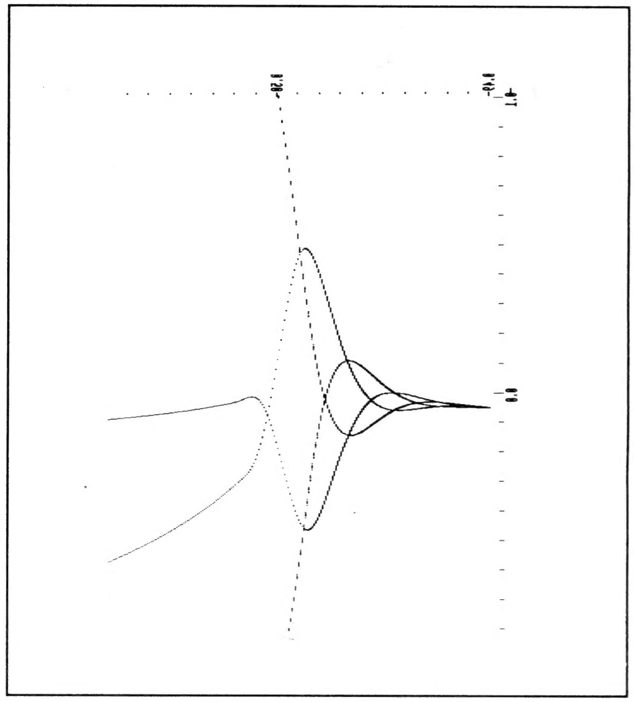


Figure 4 — Verhulst Process, Mandelbrot Variant for the Line $\Re c$ = 0.25 (700 Transients Omitted)

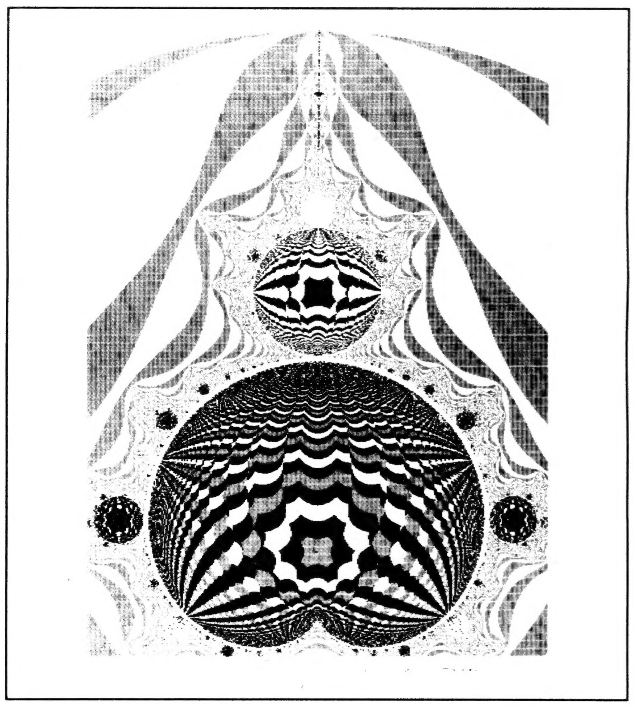


Figure 5 — Iterations to Reach Attractor 400 Max Dwell

The final map (Figure 6) is an enlargement of the interior of the cardioid, and reveals a remarkable property: the number of crenellations around each successive zone outwards increases by exactly two. Does anybody know why?

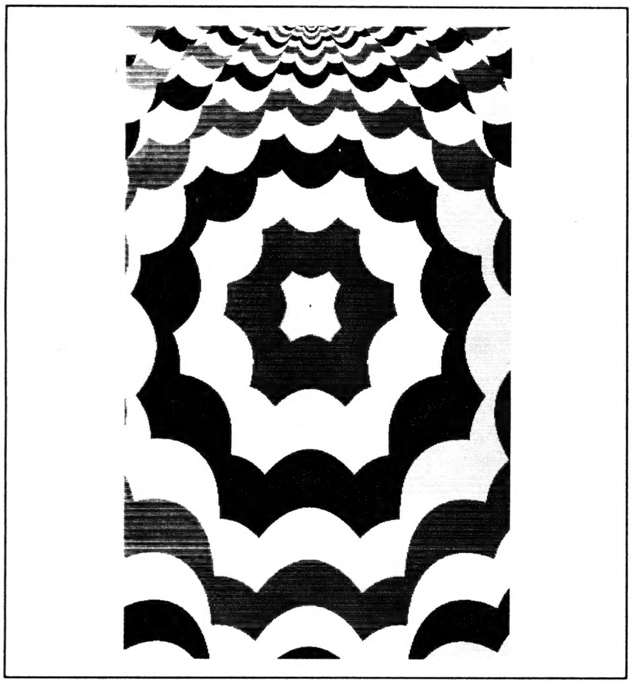


Figure 6 — Iterations to Reach Attractor 300 Max Dwell

Since writing the above I realised that the crenellations might be an artifact arising from how the program defines the proximity of a point to its attractor.

The program iterates the Mandelbrot formula for a given value of c until the value of the real part and the value of the imaginary part are each within a given (very small) distance of the corresponding real and imaginary parts of a previous iterate in the same sequence. This means that the program defines a very small square around each iterate, and looks for a previous iterate that falls within the square. By using a more complex method one could define a circle rather than a square within which to look for a sufficiently close previous iterate. If this were done, I wonder whether the crenellations would disappear?

In Figure 5, the colours have been assigned according to the dwell modulo 8. This gives different effects in the various different buds. Do these different effects relate in some way to the periods of the attractors in the buds?

THE SECANT METHOD FOR REAL AND COMPLEX VARIABLES

— John Dewey Jones

The chaotic behaviour of Newton's method for real and complex variables is well known (See for example The *Beauty of Fractals,* chapters 7 & 6 respectively). The secant method has received much less attention, though, as we shall see, its properties are at least as interesting.

Like Newton's method, the secant method is an iterative technique for finding the roots of equations. Newton's method makes a guess at the root, then constructs the tangent at the guessed value and takes the point where the tangent crosses the axis as the next guess. One drawback with this method is that it requires finding the tangent to the curve, that is, differentiation of the function. In some cases this may be difficult or impossible: the function may not be known explicitly. It may, for example, take the form of a lengthy computer program whose workings are hidden from you. Under these circumstances, we fall back on the secant method. Rather than constructing a tangent at a single point, we now make two guesses and construct the secant joining the function values at these points. The point where the secant cuts the axis gives us our next guess, and we use this together with one of the original guesses for the next iteration.

Formally, the iterative calculation is prescribed by

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

$$= \frac{f(x_n)x_{n-1} - f(x_{n-1})x_n}{f(x_n) - f(x_{n-1})} \qquad [1]$$

It can be seen from the second form of the expression (or from thinking about the geometric representation) that the calculation is symmetric in x_n and x_{n-1}.

Convergence is considered to occur when successive guesses fall within a preset limit, ε, of each other.

Image SR1 (#1029) shows the results of applying the secant method to the equation $x^4 - 1 = 0$.

The horizontal and vertical axes correspond to values of the first and second guesses respectively,

each over the range (–7,7).

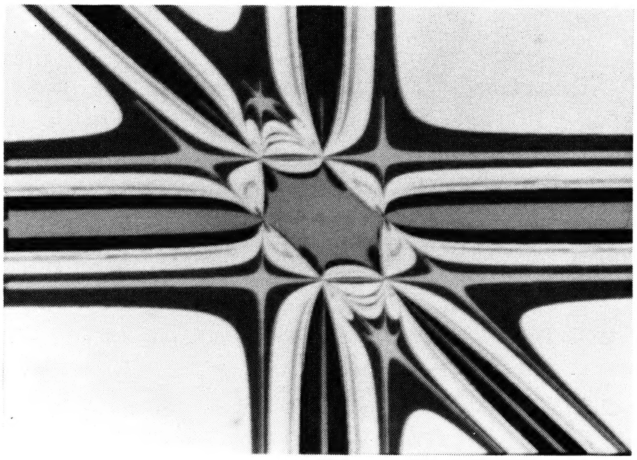


Image SR1

The colours[1] of each point have been chosen to indicate the number of iterations required to obtain convergence, pink and dark red representing the most rapid convergence. Pairs of initial guesses taken from the blue and black zones respectively never converge or converge only slowly. (This image, like those that follow, was generated on a Sun SparcStation, with resolution 1280 by 900, using Sun's 'pixrect' package to handle the bit-mapping. The application program was written in C.)

Image SR2 (#1030) was similarly generated from the equation $x^3 - 1 = 0$.

1. Professor Jones supplied images SR1-3 and SC1–7 as color slides, and his descriptions of the images refer to those slides. The set of ten slides is available as *Secant Method Slides* (#1029-1038); see the current *Amygdala Order Form* for prices and ordering information. Four of the slides appear in the slide set supplement to this issue of *Amygdala*. See THE SLIDES (S25) (Page 1) for details.

   The images appear in this article as greyscale halftones. The reader will have to use his imagination to correlate the greys with the colors referred to.

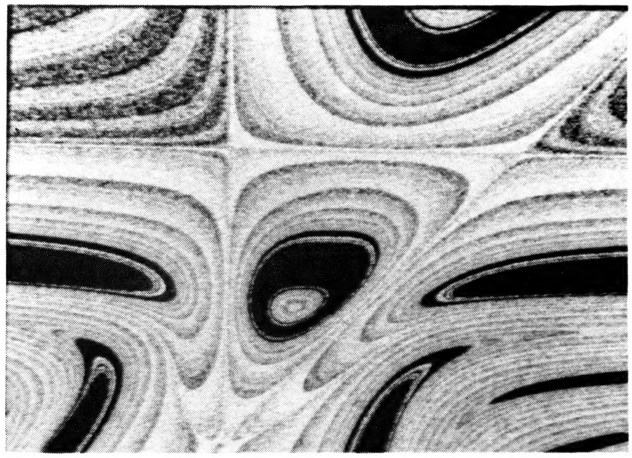


Image SR2

The limits on the axes ((–7 to 7), (–7 to 7)) are given in the legend at the top left of the slide. It is interesting to note that this image is not symmetric with respect to interchange of the axes, despite the symmetry of equation [1]. This asymmetry results from the fact that, although x_2 is determined by an expression symmetric in x_0 and x_1, x_1 is used again in the second iteration, while x_0 is not. A second curious property of this image is the mottling that appears within the regions. We found this mottling occurred while using double-precision arithmetic, and grew more pronounced as the value chosen for ε was made smaller (For Image SR3, $\varepsilon = 10^{-28}$).

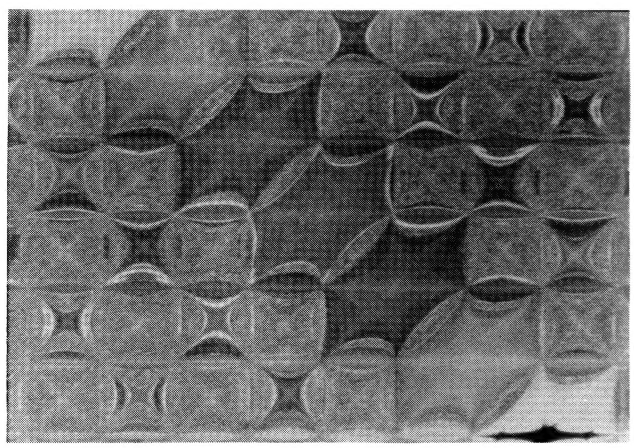


Image SR3

We had planned to carry out a systematic investigation into its cause, but before we could do so, it

stopped happening. This coincided with a change in operating system, so we suspect some subtle feature of the compiler may have been responsible. Whatever the cause, the visual effect is quite pleasing.

Image SR3 (#1031) results from applying the secant method to $\sin x = 0$.

The image represents a range of pairs of initial guesses, from (–21,–21) to (21,21), and is divided into square cells, each cell being π radians on each side. This time the hues indicate the root converged to: the equation has the infinite series of roots $(0, \pm\pi, \pm 2\pi, \ldots)$ and the degree of saturation indicates the time taken to convergence. Thus varying shades of magenta denote pairs of initial guesses converging to the root at 0, shades of indigo denote pairs converging to the roots at $\pm\pi$, and so on. White regions do not converge within the preset dwell limit of 60 iterations.

The secant method may also be used to solve equations in complex variables. Algebraically, the calculation proceeds as for the real case, though the equivalent geometric construction is more difficult to visualise as it now occurs in a space of four dimensions. The two guesses z_0 and z_1 required to start the calculation have four independent components, so the graph representing the dependence of convergence time and root on those initial guesses must be four-dimensional. Constructing a full graph with the same level of detail as the images above, that is, considering 1000 alternatives for each component of the initial guesses and iterating to a dwell limit of 64 would require about 64 trillion repetitions of the iterative step, and the results would be difficult to view on a conventional display device (though a holographic movie might come close). Rather than go to these lengths, we take a series of two-dimensional slices through the four-dimensional object. This is analogous to dissecting a biological specimen: we must wield the scalpel in such a way as to expose the inner structure of the graph, then stain it with a colour scheme that will reveal its significant details.

The images SC1-SC4 (#1032-1035) are different sections through the four-dimensional object representing the application of the secant method to $z^3 - 1 = 0$.

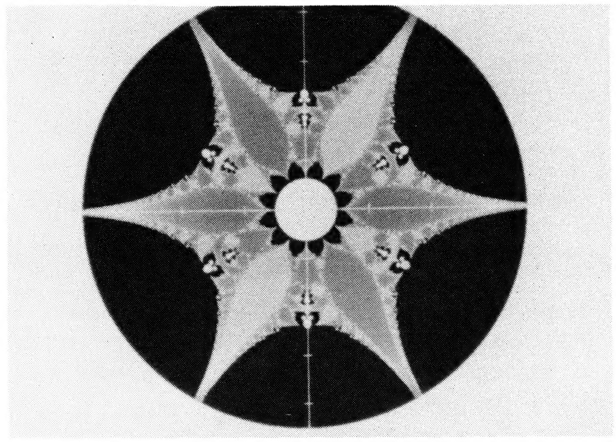


Image SC1

In SC1 (#1032), we have constrained z_0 to be zero and coloured the points representing z_1 according to the root they converge to: red (root = 1), blue (root = $\frac{-1 + i\sqrt{3}}{2}$) and green (root = $\frac{-1 - i\sqrt{3}}{2}$); as the time to convergence increases, red shades to yellow, blue shades to violet and green shades to cyan. Black regions do not converge within the preset dwell limit of 80.

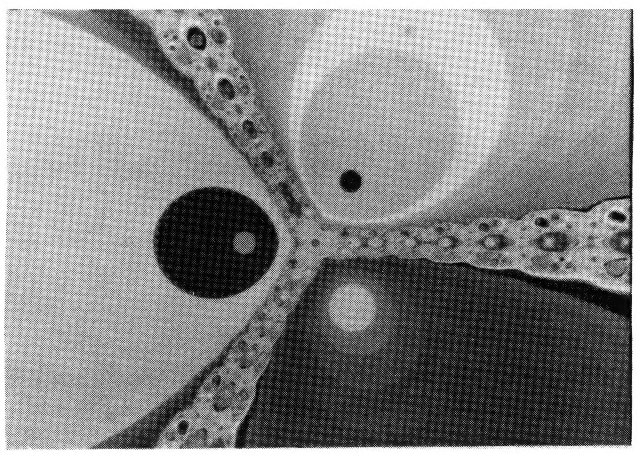


Image SC2

SC2 (#1033) is a second section through the same object. This time we have let z_0 vary over the region given by

$$|\Re(z_0)| \le 7 \qquad |\Im(z_0)| \le 7$$

and chosen z_1 to differ by a very small amount from z_0.

The three large zones with onion-like rings of colour correspond to the three roots; the pale blue shapes in the radial veins are regions which do not converge. It will be noted that this image is strikingly similar to those obtained from the complex Newton method, and this should not be surprising — as we move the two guesses closer together the secant to the curve approaches the tangent.

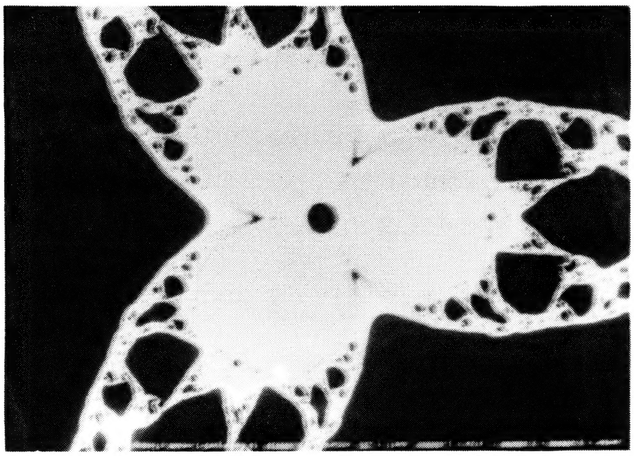
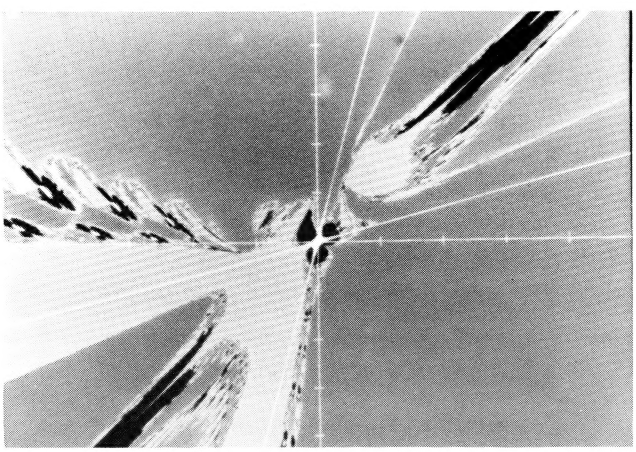


Image SC3

SC3 (#1034) shows an enlargement of the central region of SC2, lying between the limits ±1 on each axis.



SC4 (#1035) shows values of z_0 in the range

$$|\Re(z_0)| \le 7 \qquad |\Im(z_0)| \le 7$$

The corresponding values of z_1 are generated by the transformation

$$\Im(z_1) = \Re(z_0) \qquad \Re(z_1) = \Im(z_0)$$

This time we seem to have wielded the scalpel carelessly. The structure revealed is distorted and uninformative. Perhaps this is not a good plane to slice along?

But taking a similar slice through a different object — that is, interchanging the real and imaginary parts of z_0 to obtain z_1 — gives good results.

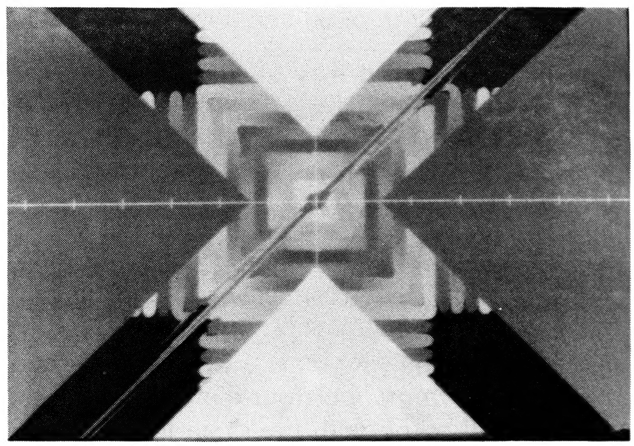


Image SC5

SC5 (#1036) is just such a slice through the object $\sin z = 0$; z_0 lies in the range

$$|\Re(z_0)| \le 110 \qquad |\Im(z_0)| \le 110$$

Different hues denote the roots $(0, \pm\pi, \pm 2\pi, \ldots)$, and the saturation of each hue varies with the number of iterations required for convergence. The white and black zones correspond to convergence to a fixed value which is not a legitimate root, and to non-convergence, respectively.

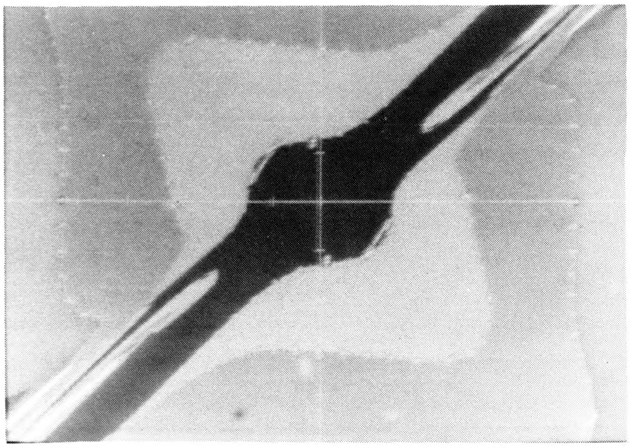


Image SC6

SC6 (#1037) is an enlargement of the central region of SC5 (the area lying within the values ±7 on each axis). We note that the basins of convergence are streaky and soft-edged. Also, there are small stitch-marks on the borders between the coloured zones.

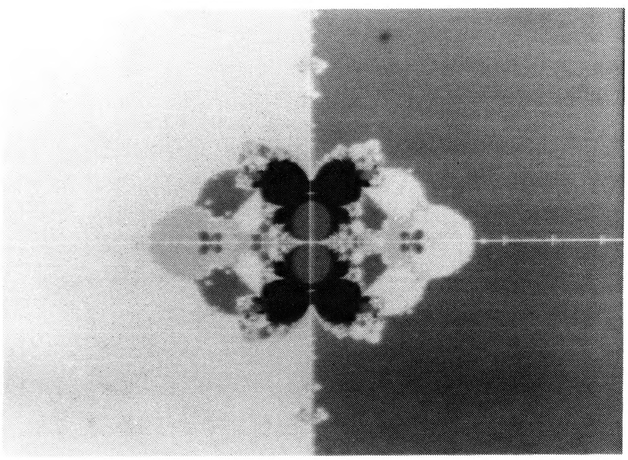


Image SC7

SC7 (#1038) shows a close-up of one of these stitch-marks, which is seen to have a complex anti-symmetric structure with black zones of non-convergence, rather similar to the shape found at the centre of SC3.

AUTHORS

Professor *John Dewey Jones* was born in England, and sometime later acquired a degree in mathematical physics. After an interlude teaching high school in Kenya, he took a doctorate in engineering and went to work for General Motors Research Laboratories, where he became interested in fractals.

He now works and teaches in the School of Engineering Science at Simon Fraser University, Canada, where he has just achieved tenure.

His current research interests include Stirling engines, applications of artificial intelligence to engineering, and fractals.

Dr. Martin Dace was fired from his post at the Theological Department of the University of Altair for his heretical views. (See *The Amygdalan Sects*, in Amy #1) He now works as a family doctor in southeast London, England. He is the co-author (with J.D. Jones) of *The Dace Julia Sets*, which appeared in Amy #10.

COMPUTERS IN ROMANIA

The Applied Computer Society (SIA), of Cluj, Romania, is anxious to publish a computer magazine in that country — and needs help!

Anyone wishing to help out with technical material, contributions of money, etc. should write to:

Marius F. Danca

ProInformatica, PO Box 524

3400 Cluj-9

ROMANIA

CIRCULATION

As of July 18, 1991 Amygdala has 518 subscribers, 194 of whom have the supplemental color slide subscription.

RENEWAL

For 63 of you subscribers out there this is the last issue of your current subscription. I urge you to use the enclosed form to renew your subscription promptly to avoid missing anything.

MERZ

Fractal software, publications, and other stuff available from individuals and small companies.

Dada was a movement of artists and writers in the early 20th century who exploited accidental and incongruous effects in their work and who programmatically challenged established canons of art, thought and morality.

Dada was originally called *Merz* — the innermost syllable of *Commerzial.*

Merz is an issue-by-issue directory of products and services provided to our readers by individuals and small companies or organizations. We will accept ads for software of educational or recreational value, pamphlets, books, videotapes, clubs, organizations, T-shirts, fractal images, calendars, etc.

To place a Bulletin Board item, send camera-ready copy to fit into one or more rectangles — or fill out your copy by hand and we will typeset it. Be sure to include all relevant information, such as address and conditions of distribution. We will accept 1x1 areas (like top right), 1x2 areas (like this) 2x2 (like the one below) or 2x5 (full page).

Ads cost 30.00 per issue per basic rectangle. There is a 10% discount for 2 or more insertions placed at the same time (not necessarily for the same issue). Send to:

Amygdala Ads • Box 219 • San Cristobal, NM 87564
Telephone: 505/586-0197

Mention Amygdala, Please!

If you place an order for a product as a result of seeing it advertised here, please let the advertiser know!

AMYGDALA, Box 219, San Cristobal, NM 87564
505/586-0197
See ORDER FORM for subscription information